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Strong regularity of matrices in general max–min algebra ☆

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Abstract

The problem of the strong regularity of a square matrix in a general max–min algebra is considered and a necessary and sufficient condition using the trapezoidal property is described. The results are valid without any restrictions on the underlying max–min algebra, concerning the density, or the boundedness. Previous results on this topic are special cases of the theorems presented in this paper.

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1. Introduction

Fuzzy relation equations are important in many applications, such as discrete dynamic systems, fuzzy control systems, or knowledge engineering, see, e.g. [8,9]. The solvability of a given fuzzy relation equation was considered in [14,15], later in [11–13]. Solving a fuzzy relation equation can be reduced to solving several linear systems of equations in fuzzy algebra.

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Max–min algebra is one of the most important fuzzy algebras. The question of unique solvability of a system of linear equations in a max–min algebra is closely connected with the strong regularity of max–min matrices. The solvability and unique solvability of linear systems in max–min algebra were studied in [1–7], where a number of interesting results are presented for special cases of max–min algebras, such as discrete, dense, bounded, or unbounded algebras. These results are completed and generalized for general max–min algebras in this work and in [10].

The aim of this paper is to present a necessary and sufficient condition for the strong regularity of a max–min matrix. We use a generalization of the notion of a trapezoidal matrix which was introduced in [2], for dense max–min algebra, and later generalized to strong trapezoidal matrix in [4], for matrices in a discrete unbounded algebra, and in [6], for the discrete bounded case. Our approach works also for general max–min algebras which are neither dense nor discrete, i.e. the results proved here are valid without any restrictions on the underlying max–min algebra, concerning the density, or the boundedness. Previous results in the above-cited papers are special cases of the theorems presented in this paper.

2. Solvability and unique solvability

By a max–min algebra $\mathscr{B}$ we mean any linearly ordered set $(\mathscr{B}, \leqslant)$ with the binary operations of maximum and minimum, denoted by $\oplus$ and $\otimes$. For any natural $n > 0$, $\mathscr{B}(n)$ denotes the set of all n-dimensional column vectors over $\mathscr{B}$, and $\mathscr{B}(m, n)$ denotes the set of all matrices of type $m \times n$ over $\mathscr{B}$. For $x, y \in \mathscr{B}(n)$, we write $x \leqslant y$, if $x_i \leqslant y_i$ holds for all $i \in N$, and we write $x < y$, if $x \leqslant y$ and $x \neq y$. A vector $x \in \mathscr{B}(n)$ is called increasing (strictly increasing), if $x_i \leqslant x_j$ holds for every $i \leqslant j$ ($x_i < x_j$ holds for every $i < j$). The matrix operations over $\mathscr{B}$ are defined with respect to $\oplus$, $\otimes$, formally in the same manner as the matrix operations over any field. In this section we consider a system of linear equations of the form

$$A \otimes x = b \tag{2.1}$$

where the matrix $A \in \mathscr{B}(m, n)$ and the vector $b \in \mathscr{B}(m)$ are given, and the vector $x \in \mathscr{B}(n)$ is unknown.

The question of solvability and unique solvability of system (2.1) was considered in [10]. The results and the notation introduced in [10] will be useful in the following section.

In general, $\mathscr{B}$ need not be bounded. We shall denote by $\mathscr{B}^\star$ the bounded algebra derived from $\mathscr{B}$ by adding the least element, or the greatest element (or both), if necessary. If $\mathscr{B}$ itself is bounded, then $\mathscr{B} = \mathscr{B}^\star$. The least element in $\mathscr{B}^\star$ will be denoted by O, the greatest one by I. To avoid the trivial case, we assume $O < I$.

Let a matrix $A \in \mathscr{B}(m, n)$ and a vector $b \in \mathscr{B}(m)$ be fixed in this section. We shall use the notation $M = \{1, 2, \ldots, m\}$, $N = \{1, 2, \ldots, n\}$. Further, we denote the solution sets

$$S^{\star}(A,b) := \{x \in \mathscr{B}^{\star}(n);\ A \otimes x = b\}$$
$$S(A,b) := \{x \in \mathscr{B}(n);\ A \otimes x = b\}$$

i.e. $S(A,b) := S^{\star}(A,b) \cap \mathscr{B}(n)$.

It was shown in [5], that the considerations may be reduced to the case when $b_i > O$ for all $i \in M$.

Lemma 2.1 [5]. *Let $i \in N$ with $b_i = O$, let us denote $N_i := \{j \in N;\ a_{ij} > O\}$. For any vector $x \in \mathscr{B}(n)$, we denote by A', b', x' the matrix and the vectors created from A, b, x by deleting the* ith *row* (*the* ith *equation*) *and all columns* (*all variables*) *with indices in N_i. Then $x \in S(A,b)$ if and only if $x' \in S(A',b')$ and $x_j = O$ for all $j \in N_i$.*

The solvability of system (2.1) is closely related to the greatest solution denoted by $\bar{x}(A,b)$, or simply by $\bar{x}$, if no error can arise. Using the notation introduced in [5] and generalized in [10], we define the vector $\bar{x}(A,b) \in \mathscr{B}^{\star}(n)$ by putting, for every $j \in N$,

$$M_j := \{i \in M;\ a_{ij} > b_i\}, \quad \bar{x}_j := \min_{\mathscr{B}^{\star}} \{b_i;\ i \in M_j\}.$$

If the matrix A and vector b are not clear from the context, we shall use a more complex notation $M_j(A,b)$ and $\bar{x}(A,b)$, instead of M_j and $\bar{x}$. The vector $\bar{x}$ is defined correctly, because the minimum in the definition is computed in a bounded algebra $\mathscr{B}^{\star}$. Therefore, every value $\bar{x}_j$ is well-defined, even in the case, when M_j is an empty set (then $\bar{x}_j = \min_{\mathscr{B}^{\star}} \emptyset = I \in \mathscr{B}^{\star}$).

Lemma 2.2 [10]. *Let $x \in S(A,b)$. Then $x \leqslant \bar{x}$ and $\bar{x} \in S^{\star}(A,b)$.*

Theorem 2.3 [10]. *Let $A \in \mathscr{B}(m,n)$, $b \in \mathscr{B}(m)$. The system $A \otimes x = b$ has a solution $x \in \mathscr{B}(n)$ if and only if $\bar{x}(A,b) \in S^{\star}(A,b)$.*

In the view of the above statements, we denote the set of all potential solutions of (2.1) by

$$\overline{S}(A,b) := \{x \in \mathscr{B}(n);\ x \leqslant \bar{x}\}.$$

Further, we denote for $i \in M$, $j \in N$

$$F_{ij} := \{x \in \overline{S}(A,b);\ a_{ij} \otimes x_j = b_i\}.$$

Similarly as above, we shall use a more complex notation $F_{ij}(A,b)$, instead of F_{ij}, if matrix A and vector b are not clear from the context. If $x \in F_{ij}$, then we say that x_j fulfills the ith equation in (2.1). Of course, it does not mean that x is a solution. This question is answered by the following lemma.

Lemma 2.4 [10]. *Let $x \in \overline{S}(A, b)$. Then the following statements are equivalent*:

(i) $x \in S(A, b)$,
(ii) $(\exists \varphi : M \to N)(\forall i \in M) x \in F_{i\varphi(i)}$.

The next notations are needed for characterization of the unique solvability. For every $j \in N$, we define

$$I_j := \left\{i \in M; a_{ij} \geqslant b_i = \bar{x}_j\right\}, \quad \mathscr{I} := \left\{I_j;\, j \in N\right\}$$
$$K_j := \left\{i \in M; a_{ij} = b_i < \bar{x}_j\right\}, \quad \mathscr{K} := \left\{K_j;\, j \in N\right\}.$$

Again, the notations I_j and K_j are only abbreviations for more complex notations $I_j(A, b)$ and $K_j(A, b)$ used where an error could arise. We may remark that our definition of the sets I_j, K_j slightly differs from the definition given in [5], but the union $I_j \cup K_j$ remains unchanged. In the original notation used in [5], assertions (i) and (ii) of the next lemma are not true.

Lemma 2.5 [10]. *Let $j \in N$. Then*

(i) $F_{ij} = \{x \in \overline{S}(A, b); x_j = \bar{x}_j\}$, *for every $i \in I_j$*,
(ii) $F_{ij} = \{x \in \overline{S}(A, b); b_i \leqslant x_j \leqslant \bar{x}_j\}$, *for every $i \in K_j$*,
(iii) $F_{ij} = \emptyset$, *for every $i \in M - (I_j \cup K_j)$*.

In [5], a necessary condition for unique solvability is given, under assumption that max–min algebra $\mathscr{B}$ is bounded. The result uses the notion of minimal covering. If S is a set and $\mathscr{C} \subseteq P(S)$ is a collection of subsets of S, we say that $\mathscr{C}$ is a covering of S, if $\bigcup \mathscr{C} = S$, and we say that a covering $\mathscr{C}$ of S is minimal, if $\bigcup(\mathscr{C} - \{C\}) \neq S$ holds for every $C \in \mathscr{C}$.

Theorem 2.6 [5]. *Let $A \in \mathscr{B}(m, n)$, $b \in \mathscr{B}(m)$, let $\mathscr{B}$ be bounded. If the system $A \otimes x = b$ has a unique solution $x \in \mathscr{B}(n)$, then the collection $\{I_j \cup \mathscr{K}_j;\, j \in N\}$ is a minimal covering of M.*

It was shown in [5], that the above condition is not sufficient. A necessary and sufficient condition for unique solvability, in a general max–min algebra $\mathscr{B}$, is presented in [10].

Theorem 2.7 [10]. *Let $A \in \mathscr{B}(m, n)$, $b \in \mathscr{B}(m)$. The system $A \otimes x = b$ has a unique solution $x \in \mathscr{B}(n)$ if and only if the collection $\mathscr{I}$ is a minimal covering of the set $M - \bigcup \mathscr{K}$.*

In the next section we shall use the following equivalent formulation of Theorem 2.7.

Theorem 2.8. *Let $A \in \mathscr{B}(m,n)$, $b \in \mathscr{B}(m)$. The system $A \otimes x = b$ has a unique solution $x \in \mathscr{B}(n)$ if and only if there is a mapping $\varphi : M \to N$ and a subset $M' \subseteq M$ such that*

(i) *restriction $\varphi|M'$ is a bijective mapping $M' \to N$;*
(ii) *for every $i \in M'$, $i \in I_{\varphi(i)} - \bigcup \{I_j \cup K_j;\ j \in N,\ j \neq \varphi(i)\}$;*
(iii) *for every $i \in M$, $i \in I_{\varphi(i)} \cup K_{\varphi(i)}$.*

If $m = n$, then $M' = M = N$ and the condition (iii) *can be left out.*

Proof. The condition (iii) is equivalent to the assertion that $\mathscr{I} \cup \mathscr{K}$ is a covering of M, i.e., that $\mathscr{I}$ is a covering of $M - \bigcup \mathscr{K}$. On the other hand, the conditions (i) and (ii) are equivalent to the assertion that the covering $\mathscr{I}$ of $M - \bigcup \mathscr{K}$ is a minimal one. □

Theorem 2.8 is a generalization of the following theorem proved in [3] for the unbounded case and extended in [5] to bounded max–min algebras under the assumption $m = n$.

Theorem 2.9 [5]. *Let $A \in \mathscr{B}(n,n)$, $b \in \mathscr{B}(n)$. The system $A \otimes x = b$ has a unique solution $x \in \mathscr{B}(n)$ if and only if there is a permutation $\varphi : N \to N$ such that*

$$a_{i\varphi(i)} \geqslant b_i > \sum\nolimits^{\oplus} \{a_{i\varphi(j)} \otimes b_j;\ j \in N - \{i\}\}$$

where the first inequality is strict for all $i \in N$ with $b_i < I$.

3. Strong regularity

A square matrix $A \in \mathscr{B}(n,n)$ in a max–min algebra $\mathscr{B}$ is strongly regular if there is $b \in \mathscr{B}(n)$ such that $A \otimes x = b$ is a uniquely solvable system of linear equations. Strong regularity has been studied in [1–6], for the special cases of a dense and of a discrete max–min algebra, and its relations to the trapezoidal property were described. Analogous results for the general case are presented in this section.

For $x \in \mathscr{B}$, the general successor $\mathrm{GS}(x)$ of x is defined by

$$\mathrm{GS}(x) := \max \{y \in \mathscr{B};\ x \leqslant y \wedge \neg(\exists z)\ x < z < y\}.$$

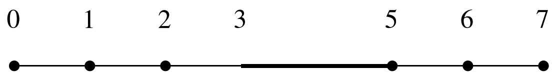
Clearly, if x is equal to the greatest element $I \in \mathscr{B}$, then $\mathrm{GS}(x) = \mathrm{GS}(I) = I$. If $x < I$, then the set $S_x = \{y \in \mathscr{B};\ y > x\}$ is non-empty. If S_x has the least element y, then $\mathrm{GS}(x) = y$, otherwise $\mathrm{GS}(x) = x$. Thus, $\mathrm{GS}(x)$ is always well defined.

We remark that in the case of a discrete max–min algebra $\mathscr{B}$, the above definition of general successor gives the same notion as in [4]. If $\mathscr{B}$ is dense, then

$\mathrm{GS}(x) = x$ for every $x \in \mathscr{B}$. Our definition applies also in the cases when $\mathscr{B}$ is neither discrete nor dense. We shall say that $x \in \mathscr{B}$ is an upper density point in $\mathscr{B}$, if $\mathrm{GS}(x) = x < I$.

The general successor of a vector $x \in \mathscr{B}(n)$ is the vector $y = \mathrm{GS}(x) \in \mathscr{B}(n)$ with $y_i = \mathrm{GS}(x_i)$, for all $i \in N$. If the vector $x \in \mathscr{B}(n)$ is increasing, then $\mathrm{GS}(x)$ is increasing as well. However, if $x \in \mathscr{B}(n)$ is strictly increasing, then $\mathrm{GS}(x)$ need not be strictly increasing, as the following example shows.

Example 1. Let us consider a max–min algebra $\mathscr{B} = \{0, 1, 2\} \cup (3, 5\rangle \cup \{6, 7\}$.



Algebra $\mathscr{B}$ is bounded by elements $O = 0$ and $I = 7$. By the above definition, $\mathrm{GS}(x) > x$ for $x = 0, 1, 5, 6$, namely $\mathrm{GS}(0) = 1$, $\mathrm{GS}(1) = 2$, $\mathrm{GS}(5) = 6$, $\mathrm{GS}(6) = 7$. For $x = 2, 7$ and for $3 < x < 5$, we have $\mathrm{GS}(x) = x$, these are the upper density points in $\mathscr{B}$.

Let us put $n = 4$ and consider vectors in $\mathscr{B}(n)$. E.g. for vectors $u = (0, 2, 4, 5)^{\mathrm{T}}$, $v = (1, 2, 6, 7)^{\mathrm{T}}$ we get $\mathrm{GS}(u) = (1, 2, 4, 6)^{\mathrm{T}}$, $\mathrm{GS}(v) = (2, 2, 7, 7)^{\mathrm{T}}$. Both vectors $u, v \in \mathscr{B}(n)$ are strictly increasing, but the general successor $\mathrm{GS}(u)$ is strictly increasing, and $\mathrm{GS}(v)$ is not.

For vectors $x, y \in \mathscr{B}(n)$, we say that x is strongly greater than y, and we write $x \sqsupset y$, when the strict inequality $x_i > y_i$ is fulfilled for every $i \in N$. Further, we say that x is almost strongly greater than y, and we write $x \sqsupseteq y$ when, for every $i \in N$, $x_i > y_i$ or $x_i = I$ holds true.

For $A \in \mathscr{B}(n, n)$, the diagonal vector $d(A) \in \mathscr{B}(n)$ and the overdiagonal maximum vector $a^{\star}(A) \in \mathscr{B}(n)$ are defined by

$$d_i(A) := a_{ii}, \quad a_i^{\star}(A) := \sum_{k=1}^{i}{}^{\oplus} \sum_{j=k+1}^{n}{}^{\oplus} a_{kj}.$$

When there will be no danger of confusion, we shall sometimes use a shorter notation d, $a^{\star}$, instead of $d(A)$, $a^{\star}(A)$.

If $A \in \mathscr{B}(n, n)$ with $n \geqslant 2$, then we define the overdiagonal delimiter $\alpha(A)$ by the following recursion. Similarly as above, we shall sometimes use a shorter notation α, instead of $\alpha(A)$. We define

$$\begin{aligned} &\alpha_1 := \mathrm{GS}(a_1^{\star}) \\ &\alpha_i := \alpha_{i-1} \oplus \mathrm{GS}(a_i^{\star}) \oplus \max\left\{\mathrm{GS}(\alpha_k);\, k < i, \alpha_k = \alpha_{i-1} \leqslant a_{ik}\right\}, \quad \text{for } i > 1. \end{aligned}$$

If $A \in \mathscr{B}(n, n)$ with $n = 1$, then we put $\alpha_1(A) = O$.

Lemma 3.1. *Let $A \in \mathscr{B}(n,n)$, $n \geqslant 2$. Then the overdiagonal delimiter $\alpha(A)$ is the least element in the partially ordered set consisting of all vectors $\alpha \in \mathscr{B}(n)$ with properties*

(i) $\alpha \geqslant \mathrm{GS}(a^{\star})$;
(ii) $i \leqslant j \Rightarrow \alpha_i \leqslant \alpha_j$;
(iii) $j < i,\ \alpha_j \leqslant a_{ij} \Rightarrow \mathrm{GS}(\alpha_j) \leqslant \alpha_i$;

for all $i, j \in N$.

Proof. First, we show that the overdiagonal delimiter $\alpha(A)$ has the desired properties. The conditions (i) and (ii) follow directly, by the recursion formulas. To verify the condition (iii), let us take $i, j \in N$, $j < i$ with $\alpha_j \leqslant a_{ij}$. If $\alpha_j < \alpha_i$, then $\mathrm{GS}(\alpha_j) \leqslant \alpha_i$. If $\alpha_j = \alpha_i$, then the monotonicity of α gives $\alpha_j = \alpha_{i-1} = \alpha_i$, i.e. $\alpha_j = \alpha_{i-1} \leqslant a_{ij}$. Then $\mathrm{GS}(\alpha_j) \leqslant \alpha_i$ holds true, by the second recursion formula.

Second, we show that vector $\alpha = \alpha(A)$ is the least element in the partially ordered set described above. Let us take a vector α' with properties (i)–(iii). We shall prove $\alpha_i \leqslant \alpha'_i$ by recursion on $i \in N$. For $i = 1$, we have $\alpha_1 = \mathrm{GS}(a_1^{\star}) \leqslant \alpha'_1$. For $i > 1$, we assume that the inequality $\alpha_j \leqslant \alpha'_j$ holds for all $j \in N$, $j < i$. Then, the properties (i) and (ii) of α' imply that (1) $\alpha_{i-1} \leqslant \alpha'_{i-1} \leqslant \alpha'_i$ and (2) $\mathrm{GS}(a_i^{\star}) \leqslant \alpha'_i$. Moreover, for any $k \in N$, $k < i$ with $\alpha_k = \alpha_{i-1} \leqslant a_{ik}$ we have (3) $\mathrm{GS}(\alpha_k) \leqslant \alpha'_i$, by the property (iii) of α'. The inequalities (1)–(3) give $\alpha_i \leqslant \alpha'_i$. $\square$

We say that the overdiagonal delimiter $\alpha(A)$ is strict in A, if for any $j, k \in N$, $j \neq k$, the equalities $\alpha_j(A) = \alpha_k(A) = I$ imply $a_{jk} < I$.

We say that a matrix $A \in \mathscr{B}(n,n)$ is generally trapezoidal, if the overdiagonal delimiter $\alpha(A)$ is strict in A and $d(A) \sqsupseteq \alpha(A)$.

Example 2. Let $\mathscr{B}$ be the max–min algebra described in Example 1. We put $n = 4$ and consider two matrices $A, B \in \mathscr{B}(n,n)$:

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 7 & 6 \\ 0 & 0 & 7 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 4 & 2 & 2 \\ 7 & 7 & 6 & 4 \\ 7 & 7 & 7 & 5 \end{bmatrix}.$$

By the above definitions, $a^{\star}(A) = (0, 0, 6, 6)^{\mathrm{T}}$ and $\mathrm{GS}(a^{\star}(A)) = (1, 1, 7, 7)^{\mathrm{T}}$. The vector $\mathrm{GS}(a^{\star}(A))$ clearly fulfills the conditions (i) and (ii) in the definition of the overdiagonal delimiter. To fulfil the condition (iii), the elements $a_{21} = 1$ and $a_{43} = 7$ must be considered. We get $\alpha(A) = (1, 2, 7, 7)^{\mathrm{T}}$. The diagonal vector $d(A) = (4, 4, 7, 7)^{\mathrm{T}}$ is almost strongly greater than the overdiagonal delimiter $\alpha(A)$, however, the matrix A is not generally trapezoidal, because the delimiter $\alpha(A)$ is not strict in A. Namely, the equalities $\alpha_3(A) = \alpha_4(A) = 7 = I$, should imply $a_{43} \neq I$, but we have $a_{43} = 7 = I$.

For the matrix B, we get $a^\star(B) = (0, 2, 4, 4)^{\mathrm{T}}$ and $\mathrm{GS}(a^\star(B)) = \alpha(B) = (1, 2, 4, 4)^{\mathrm{T}}$. The diagonal vector $d(B) = (2, 4, 6, 5)^{\mathrm{T}}$ is almost strongly greater than $\alpha(B)$ (it is even strongly greater than $\alpha(B)$), and the overdiagonal delimiter $\alpha(B)$ is strict in B. Therefore, B is generally trapezoidal.

Analogously as for the notion of a general successor, we remark that for the case of a discrete max–min algebra $\mathscr{B}$, the overdiagonal delimiter $\alpha(A)$ was defined in [4] (without any special name) and the notion of a generally trapezoidal matrix in the sense of the above definition was denoted as strongly trapezoidal. For a dense algebra $\mathscr{B}$, the overdiagonal delimiter is equal to the overdiagonal maximum vector $\alpha(A) = a^\star(A)$ and our definition of a generally trapezoidal matrix coincides with the definition of a trapezoidal matrix in a dense algebra [2,5]. Thus, the above definition is a generalization of both cases and applies also when $\mathscr{B}$ is neither discrete nor dense.

If $A \in \mathscr{B}(n, n)$ and $b \in \mathscr{B}(n)$, then we say that A is b-normal, if $i \in I_i(A, b)$ for every $i \in N$.

Theorem 3.2. *Let $A \in \mathscr{B}(n, n)$. Then the following statements are equivalent*:

(i) *A is generally trapezoidal*,
(ii) *there is an increasing vector $b \in \mathscr{B}(n)$ such that A is b-normal and the system $A \otimes x = b$ is uniquely solvable.*

The proof of Theorem 3.2 will be divided into several lemmas.

Lemma 3.3. *Let $A \in \mathscr{B}(n, n)$ be generally trapezoidal matrix. Then there is an increasing vector $b \in \mathscr{B}(n)$ such that*

(i) $d(A) \sqsupseteq b \sqsupset a^\star(A)$,
(ii) $(\forall i, j \in N)\big[(j < i \wedge b_j \leqslant a_{ij}) \Rightarrow b_j < b_i\big]$.

Proof. By assumption, A is generally trapezoidal, i.e. the overdiagonal delimiter $\alpha(A)$ is strict in A and $d(A) \sqsupseteq \alpha(A)$. First we define an increasing vector $\bar{d} \in \mathscr{B}(n)$ by putting $\bar{d}_i := \min\{d_j; i \leqslant j\}$, for every $i \in N$. We shall show that $d \geqslant \bar{d} \sqsupseteq \alpha$ holds true. The inequality $d \geqslant \bar{d}$ is trivial. To prove the second inequality, let us fix an arbitrary index $i \in N$. If $\alpha_i < I$, then, by the assumption $d(A) \sqsupseteq \alpha(A)$, we have either $d_j > \alpha_j \geqslant \alpha_i$, or $d_j = \alpha_j = I > \alpha_i$, for every $j \geqslant i$. Therefore, $\bar{d}_i = \min\{d_j; i \leqslant j\} > \alpha_i$. If $\alpha_i = I$, then, by the monotonicity of α, the equalities $d_j = \alpha_j = I$ hold true for every $j \geqslant i$, i.e. $\bar{d}_i = I = \alpha_i$. Thus, $\bar{d} \sqsupseteq \alpha$.

An increasing vector $b \in \mathscr{B}(n)$ satisfying the conditions (i) and (ii) will be defined by recursion of $k \in N$. During the recursion we assume that an increasing sequence of values b_i has been defined for all $i < k$ and the inequalities implied by conditions (i) and (ii) are fulfilled for all indices $i < k$. In addition, on some steps we assume that $b_i < \alpha_k$ for all $i < k$.

Step 1. Put the initial value $k := 1$. Then the recursion assumptions are trivially satisfied, including the additional one, and the recursion goes on by Step 2.

Step 2. It is assumed that the additional condition is satisfied when entering Step 2. Find the least $s \in N$, $s \geqslant k$, such that $\alpha_s = \mathrm{GS}(\alpha_s)$. If there is no such $s \in N$, then put formally $s = n + 1$. Then $\alpha_i < \mathrm{GS}(\alpha_i)$ holds true for every i with $k \leqslant i < s$ and we define $b_i = \alpha_i$ for all $i = k, k+1, \ldots, s-1$. By the additional assumption, we have $b_j < \alpha_k \leqslant \alpha_i = b_i$ for every $1 \leqslant j < k \leqslant i < s$, and, by the monotonicity of α, we have $b_j \leqslant \alpha_j \leqslant \alpha_i = b_i$, for every $k \leqslant j \leqslant i < s$. Thus, the values $(b_i;\, 1 \leqslant i < s)$ form an increasing sequence.

We shall verify that the sequence fulfills the inequalities implied by the conditions (i) and (ii). Let $i \in N$, $k \leqslant i < s$ be arbitrary, but fixed. Then we have $b_i = \alpha_i \geqslant \mathrm{GS}(a_i^\star) \geqslant a_i^\star$ which implies $b_i \geqslant a_i^\star$. The assumption $b_i = a_i^\star$ gives $\alpha_i = \mathrm{GS}(a_i^\star) = a_i^\star$, i.e. $\alpha_i = \mathrm{GS}(\alpha_i)$, which is in contradiction with $\alpha_i < \mathrm{GS}(\alpha_i)$. Thus, $b_i > a_i^\star$ holds true. Further, we have $d_i > \alpha_i = b_i$, or $d_i = \alpha_i = b_i = I$, in view of the assumption $d(A) \sqsupseteq \alpha(A)$.

Finally, let $j < i$, $b_j \leqslant a_{ij}$. If $j < k$, then, by the additional recursion assumption, we have $b_j < \alpha_k \leqslant \alpha_i = b_i$, i.e. $b_j < b_i$. On the other hand, if $k \leqslant j < i < s$, then, by the third property in the definition of the overdiagonal delimiter $\alpha(A)$, we have $b_j = \alpha_j < \mathrm{GS}(\alpha_j) \leqslant \alpha_i = b_i$, i.e. $b_j < b_i$.

We remark that the additional condition $b_i < \alpha_s$ need not be satisfied when leaving Step 2. Moreover, it may occur that $s = k$, and then no new values b_i are defined at this step. If $s = n + 1$, then the recursion goes to Step 4 and stops. Otherwise, the computation continues by Step 3.

Step 3. When entering this step, the additional condition is not assumed. Find the least $t \in N$, $t > s$, such that $\alpha_s < \alpha_t$. If there is no such $t \in N$, then put formally $t = n + 1$ and $\alpha_t = \bar{d}_s$. As Step 3 can be entered only from Step 2, we have $\alpha_s = \mathrm{GS}(\alpha_s)$. Thus, either α_s is an upper density point in $\mathscr{B}$, or $\alpha_s = I$. In the first case we can find a strictly increasing sequence $(b_i \in \mathscr{B};\, s \leqslant i < t)$, such that the inequalities

$$\alpha_s < b_i < \min(\alpha_t, \bar{d}_s) \leqslant \alpha_t$$

hold true for every $s \leqslant i < t$. We remark that $k \leqslant s < t$ holds true and the additional assumption $b_i < \alpha_t$, for all $i < t$, is satisfied in this case.

In the case $\alpha_s = I$, we have $d_i = \alpha_i = \alpha_s = I$ for all $i \geqslant s$ and we define the corresponding values by putting $b_i := I$.

We shall show that the values $(b_i;\, 1 \leqslant i < t)$ form an increasing sequence fulfilling the inequalities implied by conditions (i) and (ii).

If α_s is an upper density point in $\mathscr{B}$, then the inequalities implied by the conditions (i) and (ii) are trivially fulfilled, by the above definition of values $(b_i;\, s \leqslant i < t)$.

If $\alpha_s = I$, then $d_i = b_i = I$ for all $i \in N$, $i \geqslant s$, i.e. $d \sqsupseteq b$. Let $i \geqslant s$ be arbitrary, but fixed. Then the equality $a_i^\star = I$ cannot hold true, because this would imply that there exist $l, j \in N$, $s \leqslant l \leqslant i$, $l < j \leqslant n$ with $a_{lj} = I$, which would be in contradiction with the assumption that the overdiagonal delimiter α is strict in A. Therefore, $b_i = I > a_i^\star$, i.e. $b \sqsupset a^\star$. Finally, let $j < i$, $b_j \leqslant a_{ij}$. The assumption $s \leqslant j$ gives

$\alpha_j = I = b_j \leqslant a_{ij}$, i.e. $a_{ij} = I$, which again is in contradiction with the assumption that α is strict in A. Therefore, $j < s$ must hold true. Then, by the definitions in Steps 2 and 3, and by the monotonicity of α, we have $\alpha_j < I$ and $b_j < I$, which gives $b_j < I \leqslant b_i$. Thus, the condition (ii) is fulfilled, too.

At the end of Step 3, we compare the indices t, n. If $t \leqslant n$, then we put $k := t$ and go back to Step 2. This can only occur when $\alpha_s < I$, thus the additional condition is fulfilled when entering Step 2 again. If $t = n + 1$, then the recursion goes to Step 4.

Step 4. At this step, an increasing sequence $(b_i; i \in N)$ fulfilling the conditions (i) and (ii) has been found, and the computation stops. $\square$

The following two examples illustrate the recursion used for the construction of the vector b in the proof of Lemma 3.3.

Example 3. We shall use the max–min algebra $\mathscr{B}$ described in Example 1, and the matrix $B \in \mathscr{B}(n, n)$ with $n = 4$ from Example 2. We have shown there that the matrix B is generally trapezoidal, therefore the assumption of Lemma 3.3 is satisfied.

Following the proof, we begin with defining $\bar{d}_i(B) := \min\{d_j(B); i \leqslant j\}$, for every $i \in N$. We get $d(B) = (2, 4, 6, 5)^{\mathrm{T}}$ and $\bar{d}(B) = (2, 4, 5, 5)^{\mathrm{T}}$. Further, we know from Example 2, that $a^{\star}(B) = (0, 2, 4, 4)^{\mathrm{T}}$ and $\alpha(B) = (1, 2, 4, 4)^{\mathrm{T}}$.

The recursion on $k \in N$ starts at Step 1, by putting $k = 1$. In the following Step 2, the recursion finds the least $s \in N$, $s \geqslant k$, such that $\alpha_s(B) = \mathrm{GS}(\alpha_s(B))$. In our example, $s = 2$, and for every i with $k \leqslant i < s$, i.e. for $i = 1$, we define $b_i = \alpha_i(B)$. Thus, the recursion has found the first value $b_1 = \alpha_1(B) = 1$.

In Step 3, the recursion finds the least $t \in N$, $t > s$, such that $\alpha_s(B) < \alpha_t(B)$. Here we get $t = 3$, and we have to find a strictly increasing sequence $(b_i \in \mathscr{B}; s \leqslant i < t)$, such that the inequalities

$$\alpha_s(B) < b_i < \min\left(\alpha_t(B), \bar{d}_s(B)\right) \leqslant \alpha_t(B)$$

hold true for every $s \leqslant i < t$. Thus, we choose, e.g. $b_2 = 3.1$. As $t \leqslant n$, the recursion puts $k = t = 3$ and goes back to Step 2, where we get $s = k = 3$. Therefore, no new value b_i is defined at this moment, and the recursion goes on to Step 3.

In the second visit of Step 3, the recursion looks for the least $t \in N$, $t > s$, such that $\alpha_s(B) < \alpha_t(B)$. As there is no such $t \in N$, the recursion puts formally $t = n + 1 = 5$ and $\alpha_t(B) = \bar{d}_s(B) = \bar{d}_3(B) = 5$. Again, we have to find a strictly increasing sequence $(b_i \in \mathscr{B}; s \leqslant i < t)$, such that the inequalities

$$\alpha_s(B) < b_i < \min\left(\alpha_t(B), \bar{d}_s(B)\right) \leqslant \alpha_t(B)$$

will hold true for every $s \leqslant i < t$, i.e. for $i = 3, 4$. We can choose e.g. $b_3 = 4.1$, $b_4 = 4.2$. Then the recursion goes to Step 4 and stops with the output $b = (1, 3.1, 4.1, 4.2)^{\mathrm{T}}$.

Remark 3.1. We may notice that in the above example, the overdiagonal delimiter $\alpha(B)$ is not a suitable candidate for the vector b, because none of the conditions (i) and (ii) would be satisfied with the choice $b = \alpha(B)$.

Example 4. In this example we use the max–min algebra $\mathscr{B}$ from Example 1, and the matrix $A \in \mathscr{B}(n, n)$ with $n = 4$ from Example 2. As we have seen there, the matrix A is not generally trapezoidal, because the overdiagonal delimiter $\alpha(A)$ is not strict in A. By applying the recursion from the proof of Lemma 3.3, we shall demonstrate that the strictness of $\alpha(A)$ plays an important role and cannot be left out.

Again, we begin with defining $\bar{d}_i(A) := \min\{d_j(A); i \leqslant j\}$, for every $i \in N$. We get $d(A) = \bar{d}(A) = (4, 4, 7, 7)^{\mathrm{T}}$. In Example 2, we have computed $a^{\star}(A) = (0, 0, 6, 6)^{\mathrm{T}}$ and $\alpha(A) = (1, 2, 7, 7)^{\mathrm{T}}$.

In Step 1, the recursion puts $k = 1$ and goes to Step 2. There the recursion finds the least $s \in N$, $s \geqslant k$, such that $\alpha_s(A) = \mathrm{GS}(\alpha_s(A))$. Similarly as in the previous example, the value $s = 2$ is found, and the first value $b_1 = \alpha_1(A) = 1$ is defined.

In Step 3, the recursion looks for the least $t \in N$, $t > s$, such that $\alpha_s(A) < \alpha_t(A)$. The value $t = 3$ is found and b_2 must belong to the open interval $(\alpha_s, \min(\alpha_t, \bar{d}_s)) = (2, \min(\alpha_3, \bar{d}_2)) = (2, \min(7, 4)) = (2, 4)$, e.g. $b_2 = 3.5$ can be chosen. We may notice that, by this definition, the condition (ii) is fulfilled for $i = 2$, $j = 1$. This is a consequence of the condition (iii) in the definition of the overdiagonal delimiter $\alpha(A)$. As $t = 3 \leqslant n = 4$, the recursion puts $k := t = 3$ and goes back to Step 2.

In the second visit of Step 2, we get $s = k = 3$, i.e. no new value b_i is defined and the recursion continues by Step 3. There, the recursion looks for the least $t \in N$, $t > s = 3$, such that $\alpha_s(A) < \alpha_t(A)$. There is no such $t \in N$, therefore the recursion puts formally $t = n + 1 = 5$ and $\alpha_t(A) = \bar{d}_s(A) = \bar{d}_3(A) = 7$. As $\alpha_s(A) = I = 7$ at this step, the recursion puts $b_i = I$ for every $s \leqslant i < t$, i.e. $b_3 = b_4 = 7$. Then the recursion goes to Step 4 and stops with the output $b = (1, 3.5, 7, 7)^{\mathrm{T}}$.

The condition (i) is satisfied by this output, because $d(A) = (4, 4, 7, 7)^{\mathrm{T}} \sqsupseteq b = (1, 3.5, 7, 7)^{\mathrm{T}} \sqsupset a^{\star}(A) = (0, 0, 6, 6)^{\mathrm{T}}$. However, the condition (ii) does not hold for $i = 4$, $j = 3$, which is caused by the fact that $\alpha(A)$ is not strict in A.

Lemma 3.4. *Let $A \in \mathscr{B}(n, n)$ be a generally trapezoidal matrix, let $b \in \mathscr{B}(n)$ be an increasing vector fulfilling the conditions* (i) *and* (ii) *from Lemma* 3.3. *Then A is b-normal.*

Proof. We have to show that $i \in I_i(A, b)$ holds for every $i \in N$. Let $i \in N$ be fixed. By the definition of $I_i(A, b)$, we have to prove that $a_{ii} \geqslant b_i = \bar{x}_i$. By condition (i), we have $d(A) \sqsupseteq b$. We remind that $d = d(A)$ is the diagonal vector in A, i.e. $a_{ii} = d_i$. We shall consider two cases.

Case 1. If $a_{ii} > b_i$ holds, then $M_i(A, b) \neq \emptyset$, with $i \in M_i(A, b)$ and $\bar{x}_i = \min\{b_k; k \in M_i\}$. For $k < i$ we have $b_k > a_k^{\star} \geqslant a_{ki}$, which implies $k \notin M_i$, in view of the definition of M_i. Therefore, $\bar{x}_i = \min\{b_k; k \geqslant i, k \in M_i\}$. This gives $b_i = \bar{x}_i$, because b is increasing and $i \in M_i$. We have shown that $a_{ii} \geqslant b_i = \bar{x}_i$, i.e. $i \in I_i(A, b)$.

Case 2. If $a_{ii} = b_i = I$, then, by the definition of M_i, we have $M_i := \{k \in M; a_{ki} > b_k\}$. Let us take $k \in M_i$, then we have $b_k < a_{ki} \leqslant I$, which implies $k < i$, by the monotonicity of b. Further, by condition (i) we get $b_k > a_k^\star \geqslant a_{ki}$, thus $k \notin M_i$, a contradiction. We have shown that $M_i = \emptyset$. Therefore, $\bar{x}_i = I = b_i$ holds true, and we again have $a_{ii} \geqslant b_i = \bar{x}_i$, i.e. $i \in I_i(A, b)$. □

Lemma 3.5. *Let $A \in \mathscr{B}(n, n)$ be generally trapezoidal matrix, let $b \in \mathscr{B}(n)$ be an increasing vector fulfilling the conditions* (i) *and* (ii) *from Lemma* 3.3. *Then the system $A \otimes x = b$ is uniquely solvable.*

Proof. For all $i \in N$, as $I_i \cap K_i = \emptyset$, we have $i \in I_i$ and $i \notin K_i$, by Lemma 3.4. In the view of Theorem 2.8, where we take as φ the identical mapping on N, it is sufficient to show that $i \notin I_j \cup K_j$ for every $j \neq i$. Let us consider three cases.

Case 1. If $i < j$, then the inequalities $b_i > a_i^\star \geqslant a_{ij}$ hold true, which imply $i \notin I_j \cup K_j$.

Case 2. If $j < i$ and $b_j \leqslant a_{ij}$, then we get $b_j < b_i$, by condition (ii). Moreover, as a consequence of Lemma 3.4, we have $j \in I_j$, i.e. $\bar{x}_j = b_j$. Therefore, $\bar{x}_j < b_i$ and $i \notin I_j \cup K_j$.

Case 3. Finally, if $j < i$ and $b_j > a_{ij}$, then $b_i \geqslant b_j > a_{ij}$, which gives $i \notin I_j \cup K_j$. □

Lemma 3.6. *Let $A \in \mathscr{B}(n, n)$ and let $b \in \mathscr{B}(n)$ be an increasing vector such that A is b-normal and the system $A \otimes x = b$ is uniquely solvable. Then the vector b fulfills the conditions* (i) *and* (ii) *from Lemma* 3.3.

Proof. We shall prove first that $d(A) \sqsupseteq b$. By assumption $|S(A, b)| = 1$ and by the b-normality of A, we have $i \in I_i$ and $i \notin I_j \cup K_j$ for every $i, j \in N$, $i \neq j$. Let $i \in N$ be arbitrary, but fixed.

If $M_i = \emptyset$, then we have $\bar{x}_i = I$ and the assumption $i \in I_i$ implies $a_{ii} \geqslant b_i = \bar{x}_i = I$, i.e. $a_{ii} = b_i = I$. If $M_i \neq \emptyset$, then there is $j \in M_i$ such that $\bar{x}_i = b_j$. Thus, we have $a_{ji} > b_j = \bar{x}_i$, i.e. $j \in I_i$. By assumption $|S(A, b)| = 1$ and by the b-normality of A, $j \in I_i$ cannot hold for $j \neq i$, therefore $j = i$, i.e. $a_{ii} > b_i$. As $i \in N$ was chosen arbitrarily, we have shown that $d(A) \sqsupseteq b$.

To prove the inequality $b \sqsupset a^\star(A)$, let us fix an index $i \in N$ and consider any $j \in N$, $j > i$. If we assume $a_{ij} > b_i$, then we get $i \in M_j$ and, by the monotonicity of b, $\bar{x}_j \leqslant b_i \leqslant b_j = \bar{x}_j$, i.e. $\bar{x}_j = b_i$. This gives $i \in I_j$, which is in contradiction with the assumption $|S(A, b)| = 1$. Therefore, $a_{ij} \leqslant b_i$ must hold. The equality $a_{ij} = b_i$ implies $b_i \leqslant b_j = \bar{x}_j$, i.e. $i \in I_j \cup K_j$, which again leads to contradiction. Thus, we have proved that $a_{ij} < b_i$ must be true, whenever $j > i$. Further, the monotonicity of b gives $b_i \geqslant b_k > a_{kj}$ for every $k \leqslant i$, $j > k$. Thus, $b_i > a_i^\star$ for arbitrary $i \in N$, i.e. $b \sqsupset a^\star(A)$. By this, the condition (i) is proved.

To prove the condition (ii), let us fix indices $i, j \in N$ with $i < j$, $b_i \leqslant a_{ji}$. If we assume $b_i = b_j$, then the equality $b_i = \bar{x}_i$ gives $a_{ji} \geqslant b_j = \bar{x}_i$, i.e. $j \in I_i$, which is a contradiction. Therefore, $b_i \neq b_j$ must be true, and by the monotonicity of b, the inequality $b_i < b_j$ holds true. □

Lemma 3.7. *Let $A \in \mathscr{B}(n, n)$ and let there be an increasing vector $b \in \mathscr{B}(n)$ fulfilling the conditions* (i) *and* (ii) *from Lemma* 3.3. *Then A is generally trapezoidal.*

Proof. Let $b \in \mathscr{B}(n)$ be a fixed increasing vector satisfying the conditions (i) and (ii) from Lemma 3.3. It is easy to see that b satisfies the conditions put on the vector α in the definition of the overdiagonal delimiter $\alpha(A)$. Thus, $b \geqslant \alpha(A)$, which gives $d(A) \sqsupseteq b \geqslant \alpha(A)$, i.e. $d(A) \sqsupseteq \alpha(A)$.

Further, we shall show that $\alpha(A)$ is strict in A. Let us suppose that $\alpha_j(A) = \alpha_k(A) = I$ and $a_{jk} = I$ for some $j, k \in N$, $j \neq k$. Then $b_j = b_k = I$ must hold, too. If $j < k$, then we have $a_j^\star = I$, which is in contradiction with $b_j > a_j^\star$, according to the condition (i). If $k < j$, then we have a contradiction with the condition (ii). Thus, the overdiagonal delimiter $\alpha(A)$ is strict, which completes the proof. □

The next theorem is a generalization of Theorem 13 in [5]. The original theorem was proved by Cechlárová for $\mathscr{B}$ on the closed real interval $\langle 0, 1\rangle$. Theorem 3.8 holds true for every max–min algebra $\mathscr{B}$, without any limitations concerning the density, the discretness, or the existence of bounds in $\mathscr{B}$.

The theorem uses the following notation. If $A \in \mathscr{B}(n, n)$ is a square matrix and φ, ψ are permutations on N, then $A_{\varphi\psi} \in \mathscr{B}(n, n)$ denotes the result of applying the permutation φ to the rows and the permutation ψ to the columns of the matrix A. We say that matrices A, B are equivalent if there are permutations φ, ψ, such that $B = A_{\varphi\psi}$, i.e. $a_{ij} = b_{\varphi(i)\psi(j)}$ for every $i, j \in N$.

Theorem 3.8. *Let $A \in \mathscr{B}(n, n)$. Then the following statements are equivalent*:

(i) *A is strongly regular*,
(ii) *A is equivalent to a generally trapezoidal matrix*, *i.e. there are permutations φ, ψ such that $A_{\varphi\psi}$ is generally trapezoidal.*

Proof. (i) ⇒ (ii). Let $A \in \mathscr{B}(n, n)$ be a strongly regular matrix, i.e. let there exist a vector $b \in \mathscr{B}(n)$ such that the system $A \otimes x = b$ is uniquely solvable. After permuting the equations in the system by any permutation φ, we get a system $A_{\varphi\varepsilon} \otimes x = b'$ (ε denotes the identical permutation on N), which is uniquely solvable, too. The permutation φ can be chosen in such a way that the right-hand side vector b' is increasing. Theorem 2.8 implies that there is a permutation ψ of variables, i.e. of the columns of $A_{\varphi\varepsilon}$, such that $i \in I_i$ $(A_{\varphi\psi}, b')$ holds for all $i \in N$. Thus, A is b'-normal and, by Theorem 3.2, A is generally trapezoidal.

(ii) ⇒ (i). Let $A \in \mathscr{B}(n, n)$ be equivalent to a generally trapezoidal matrix $A_{\varphi\psi}$. By Theorem 3.2, there is an increasing vector $b' \in \mathscr{B}(n)$ such that $A_{\varphi\psi}$ is b'-normal and the system $A_{\varphi\psi} \otimes x = b'$ is uniquely solvable. Applying the inverse permutation ψ^{-1} to the variables and the inverse permutation φ^{-1} to the equations in the system, we get a system $A \otimes x = b$, which is uniquely solvable, too. Thus, A is strongly regular. □

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